Computers and Mathematics with Applications 63 (2012) 1279–1286

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence and uniqueness of common fixed points for two multivalued operators in ordered metric spaces

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ARTICLE INFO

Article history:
Received 18 April 2011
Received in revised form 8 December 2011
Accepted 8 December 2011

Keywords:
Multivalued operator
Common fixed points
Ordered metric space
Hausdorff distance

ABSTRACT

In this paper, we discuss some new fixed point theorems for a pair of multivalued operators which satisfy weakly generalized contractive conditions. Our results are the extension and improvement of corresponding results of [J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Analysis, 71 (2009) 3403–3410] and [X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. 333 (2007) 780–786]. Finally, some examples are given to illustrate the usability of our results.

1. Introduction

It is well known that the contractive-type conditions are very important in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach–Caccioppoli theorem, published for the first time in 1922 in [1] and also found in [2]. Then Kannan analyzed a substantially new type of contractive condition in [3]. Nadler in [4] extended the contraction into multivalued mappings and obtained the existence of fixed points. Since then there have been many theorems dealing with mappings satisfying various types of contractive inequality, we refer to [5–19] and references therein. Very recently results of common fixed points for a pair of single-valued operators were obtained by applying various types of contractive conditions, we refer to [20–24]. Moreover, in [25–28] authors considered the analogy of multivalued mappings. For example, in [26], the existence of common fixed points for multivalued mappings was also considered recently by applying the monotone method in ordered Banach spaces. However, as far as our knowledge, few corresponding results of common fixed points for multivalued operators satisfying generalized contractive conditions are concerned (see [27,28]). The purpose of the present paper is to establish the common fixed point theorems for weakly contractive multivalued operators in ordered complete metric spaces. The weakly contractive single-valued maps were first defined by Alber and Guerre–Delabriere in [29]. Here we give a brief description of the basic known notions.

Let $(E, \|\cdot\|)$ be a Banach space, a selfmap F of E is said to satisfy the Banach contraction principle if there exists a constant k with $0 \le k < 1$ such that, for $x, y \in E$,

$$\|Fx - Fy\| \le k\|x - y\|.$$

As noted in the introduction of [29], this inequality can be written in the form

$$\|Fx - Fy\| \le \|x - y\| - q\|x - y\|,$$

where $k = 1 - q$ with $q \in (0, 1]$. The extension of the above inequality in the context of Banach spaces to what we called weakly contractive maps is a natural one. A selfmap F of E is said to be weakly contractive if

$$\|Fx - Fy\| \le \|x - y\| - \psi(\|x - y\|)$$

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doi:10.1016/j.camwa.2011.12.021

for every $x, y \in E$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$ is a continuous and nondecreasing function such that it is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$ (it is clear that φ needs to satisfy $\varphi(t) \le t$ for $t > 0$). [30] extended the notion to a metric space E, that is, a map $F : E \to E$ is said to be weakly contractive if

$$d(Fx, Fy) \le d(x, y) - \psi(d(x, y))$$

for all $x, y \in E$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the above mentioned conditions.

Let $(E, d, \le)$ denote an ordered complete metric space with a partial order $\le$ and distance $d(\cdot, \cdot)$. Let $d = \sup\{d(x, y) : x, y \in E\}$. Set $a = d$ if $d = \infty$ and $a > d$ if $d < \infty$. Moreover, [16] extended the notion in [30] to the weaker contraction for the multivalued operators, namely, the multivalued mapping $G : E \to 2^E$, $f \in \mathscr{F}[0, a)$ and $\varphi \in \Phi[0, f(a - 0))$ satisfy

$$f(H_d(Gx, Gy)) \le f(d(x, y)) - \varphi(f(d(x, y)))$$

for all $x, y \in E$ with x and y comparable. Then G is called a weakly generalized contraction with respect to f and φ (for the notations appear here we refers to the below definitions).

In this paper we will define an analogical weakly contractive type condition for two multivalued maps. Moreover, we will obtain some results which are also new even to the single-valued case of operator equations.

The rest of the paper is organized as follows. Section 2 deals with the preliminaries needed in the sequel. Section 3 establishes the main fixed point theorems and some corollaries in the applicable form to differential and integral inclusions and equations. To show the applicability of our results, in Section 4 we discusses several examples.

2. Preliminaries

In this paper, unless otherwise mentioned, let $(E, d, \le)$ denote an ordered complete metric space with a partial order $\le$ and distance $d(\cdot, \cdot)$. Let 2^E denote the family consisting of all nonempty subsets of E. The following hypothesis in E will be applied:

(H1) If $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x$, then $x_n \le x$ (resp. $x_n \ge x$) for all $n \in \mathbb{N}$.

We define the Hausdorff pseudometric in 2^E by $H_d : 2^E \times 2^E \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(C, D) = \max\left\{\sup_{a\in C} d(a, D), \sup_{b\in D} d(C, b)\right\},$$

where $d(C, b) = \inf_{a\in C} d(a, b)$, $d(a, D) = \inf_{b\in D} d(a, b)$.

Definition 2.1 (*[16]*)**.** Let E be a metric space. A subset $D \subset E$ is said to be approximative if the multivalued mapping

$$\mathscr{P}_D(x) = \{y \in D : d(x, y) = d(D, x)\}, \quad \forall x \in E$$

has nonempty values.

The multivalued mapping $G : E \to 2^E$ is said to have approximative values, AV for short, if Gx is approximative for each $x \in E$.

The multivalued mapping $G : E \to 2^2$ is said to have comparable approximative values, CAV for short, if G has approximative values and, for each $z \in E$, there exists $y \in \mathscr{P}_{Gz}(x)$ such that y is comparable to z.

The multivalued mapping $G : E \to 2^E$ is said to have upper comparable approximative values, UCAV, for short (resp. lower comparable approximative values, LCAV for short) if G has approximative values and, for each $z \in E$, there exists $y \in \mathscr{P}_{Gz}(x)$ such that $y \ge z$ (resp. $y \le z$).

It is clear that G has approximative values if it has compact values. In addition, if G is single-valued, then UCAV (LCAV) means that $Gx \ge x$ ($Gx \le x$) for $x \in E$.

Definition 2.2. The multivalued mapping G is said to have a fixed point if there is $x \in E$ such that $x \in Gx$.

In what follows, we give an analogy of the contraction which is called the weakly generalized contractive type condition for multivalued mappings which will play an important role in this sequel. To this end, we first introduce the following functions.

Let $a \in (0, \infty]$, $R_a^+ = [0, a)$. Let $f : R_a^+ \to \mathbb{R}$ satisfy

(i) $f(0) = 0$ and $f(t) > 0$ for each $t \in (0, a)$;
(ii) f is nondecreasing on R_a^+;
(iii) f is continuous.
(iv) $f(t + s) \le f(t) + f(s)$ for $s, t \in R_a^+$.

For examples of such function f we refer to [15]. Define $\mathscr{F}[0, a) = \{f | f$ satisfies (i)–(iv) above$\}$. It is easy to see that $\lim_{n\to\infty} f(t_n) = 0$ for $t_n \in R_a^+$, then $\lim_{n\to\infty} t_n = 0$ if $f \in \mathscr{F}[0, a)$.

Let $a \in (0, \infty]$, $\varphi : R_a^+ \to \mathbb{R}_+$ satisfy

(i) $\varphi(0) = 0$ and $\varphi(t) > 0$ for each $t \in (0, a)$.

(ii) φ is right lower semi-continuous, i.e. for any nonnegative nonincreasing sequence $\{r_n\}$, $\liminf_{n\to\infty} \varphi(r_n) \geq \varphi(r)$, provided $\lim_{n\to\infty} r_n = r$.
(iii) For any sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n = 0$, there exist $a \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $\varphi(r_n) \geq ar_n$ for each $n \geq n_0$.

Define $\Phi[0, a) = \{\varphi : \varphi$ satisfies (i)–(iii) above$\}$.

Definition 2.3. Let E be a metric space and let $d = \sup\{d(x, y) : x, y \in E\}$. Set $a = d$ if $d = \infty$ and $a > d$ if $d < \infty$. Suppose that the multivalued mappings $T, S : E \to 2^E$, $f \in \mathscr{F}[0, a)$ and $\varphi \in \Phi[0, f(a-0))$ satisfy

$$f(H_d(Tx, Sy)) \leq f(M(x, y)) - \varphi(f(M(x, y)))$$

for all $x, y \in E$ with x and y comparable, where

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}(d(Tx, y) + d(Sy, x))\right\}.$$

Then we say that T and S satisfy weakly generalized contraction with respect to f and φ.

Remark 2.4. Let E be a Banach space with the norm $\|\cdot\|$ and the metric $d(\cdot,\cdot)$ generated by it. In Definition 2.3, let $f(t) = t$ and $T = S = G$, then

$$H_d(Gx, Gy) \leq M(x, y) - \varphi(M(x, y))$$

for all $x, y \in E$ with x and y comparable. This is a immediate extension from single-valued into multivalued maps of [30,23]. Here, we omit the hypotheses of the continuity and monotonicity of φ and $\lim_{t\to\infty} \psi(t) = \infty$ by contrast with [29]. So the weakly generalized contraction is a extension and improvement of notions in[16,29,30].

In this sequel we shall also apply the following notions.

Definition 2.5. For two subset X, Y of E, we mark $X \leq_r Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$ and $X \leq Y$ if each $x \in X$ and each $y \in Y$ imply that $x \leq y$.

A multivalued mapping $G : E \to 2^E$ is said to be r-nondecreasing (r-nonincreasing) if $x \leq y$ implies that $Gx \leq_r Gy$ ($Gy \leq_r Gx$) for all $x, y \in E$. G is said to be r-monotone if G is r-nondecreasing or r-nonincreasing.

The notion of nondecreasing (nonincreasing) is similarly defined by writing $\leq$ instead of the notation $\leq_r$.

3. Main results

In this section, we shall present the existence and uniqueness of common fixed points for two multivalued mappings on ordered complete metric spaces.

Theorem 3.1. *Let E satisfy the hypothesis* (H1). *Suppose that the multivalued mappings T, S have UCAV and satisfy the weakly generalized contraction with respect to given $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a-0))$, then T, S have a common fixed point $x^* \in E$. Further, for each $x_0 \in E$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T and S.*

Proof. We first prove that any fixed point of T is also a fixed point of S and conversely. If $x^* \in Tx^*$ but $x^* \notin Sx^*$. Since Sx^* is approximate, $d(Sx^*, x^*) > 0$. From

$$M(x^*, x^*) = \max\left\{d(x^*, x^*), d(Tx^*, x^*), d(Sx^*, x^*), \frac{1}{2}(d(Tx^*, x^*) + d(Sx^*, x^*))\right\} = d(Sx^*, x^*),$$

it follows that

$$\begin{aligned} f(d(Sx^*, x^*)) &\leq f(H_d(Tx^*, Sx^*)) \leq f(M(x^*, x^*)) - \varphi(f(M(x^*, x^*))) \\ &< f(M(Sx^*, x^*)) = f(d(Sx^*, x^*)). \end{aligned}$$

This is a contradiction, so $x^* \in Sx^*$. As the same process we also can get if $x^* \in Sx^*$ then $x^* \in Tx^*$.

Given $x_0 \in E$, if $x_0 \in Tx_0$ our proof is complete. Otherwise, from the fact that Tx_0 has UCAV it follows there exists $x_1 \in Tx_0$ with $x_1 \neq x_0$ and $x_1 \geq x_0$ such that $d(x_0, x_1) = \inf_{x\in Tx_0} d(x, x_0) = d(Tx_0, x_0)$. If $x_1 \in Sx_1$ our proof is complete. Otherwise, from the fact that Sx_1 has UCAV it follows there exists $x_2 \in Sx_1$ with $x_2 \neq x_1$ and $x_2 \geq x_1$ such that $d(x_1, x_2) = \inf_{x\in Sx_1} d(x, x_1) = d(Sx_1, x_1)$. We continue the procedure of constructing x_n inductively, that is, either $x_{2n+1} \in Tx_{2n}$ or $x_{2n+2} \in Sx_{2n+1}$, then our proof is complete; or there exist $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ with $x_n \neq x_{n-1}$ and $x_n \geq x_{n-1}$ such that

$$\begin{cases} d(Tx_{2n}, x_{2n}) = d(x_{2n+1}, x_{2n}), \\ d(Sx_{2n+1}, x_{2n+1}) = d(x_{2n+2}, x_{2n+1}), \quad (n = 0, 1, 2, \ldots). \end{cases} \tag{1}$$

It is easy see that $M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(Tx_{2n}, x_{2n}), d(Sx_{2n+1}, x_{2n+1}), \frac{1}{2}(d(Tx_{2n}, x_{2n+1}) + d(Sx_{2n+1}, x_{2n}))\} = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+2}, x_{2n+1})\}$. If $d(x_{2n+2}, x_{2n+1}) \geq d(x_{2n}, x_{2n+1})$, then

$$\begin{aligned} f(d(x_{2n+2}, x_{2n+1})) &= f(d(Sx_{2n+1}, x_{2n+1})) \leq f(H_d(Tx_{2n}, Sx_{2n+1})) \\ &\leq f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1}))) \\ &< f(M(x_{2n}, x_{2n+1})) = f(d(x_{2n+2}, x_{2n+1})). \end{aligned}$$

This is a contradiction. So we have

$$d(x_{2n+2}, x_{2n+1}) < d(x_{2n}, x_{2n+1}) = M(x_{2n}, x_{2n+1}). \tag{2}$$

Take the same proceeding we have

$$d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1}) = M(x_{2n}, x_{2n-1}). \tag{3}$$

Put $\rho_n = d(x_{n-1}, x_n)$. Then (2) and (3) guarantee that $\{\rho_n\}$ is a nonnegative nonincreasing sequence and hence possesses a limit ρ^*. If $\rho^* > 0$, from the assumption (ii) of φ, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi(f(\rho_n)) \geq \varphi(f(\rho^*)) > 0 \quad \text{for all } n > n_0.$$

In addition, we have

$$f(\rho_n) \leq f(\rho_{n-1}) - \varphi(f(\rho_{n-1})) \leq f(\rho_{n-1}) - \varphi(f(\rho^*)).$$

Take limit when $n \to \infty$, from the assumptions about f and φ it follows that

$$f(\rho^*) \leq f(\rho^*) - \varphi(f(\rho^*)) < f(\rho^*).$$

This is a contradiction and hence we get $\rho^* = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. On the one hand, since $\lim_{n\to\infty} f(d(x_n, x_{n-1})) = 0$, from the assumption (iii) of φ there exists $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that

$$\varphi(f(d(x_n, x_{n-1}))) \geq af(d(x_n, x_{n-1})) \quad \text{for all } n > n_0.$$

On the other hand, for any given $\varepsilon > 0$, we can choose $\delta > 0$ to be small enough such that $f(\delta) < \frac{a}{1-a} f(\varepsilon)$. Moreover, there exists n_1 such that $d(x_{n+1}, x_n) \leq \delta$ for each $n \geq n_1$. On the other hand, $d(Tx_{2n}, x_{2n}) \leq \sup_{x \in Sx_{2n-1}} d(Tx_{2n}, x) \leq H_d(Tx_{2n}, Sx_{2n-1})$, therefore,

$$d(x_{2n+1}, x_{2n}) \leq H_d(Tx_{2n}, Sx_{2n-1}) \quad \text{for } n = 1, 2, 3, \ldots. \tag{4}$$

Similarly, we have

$$d(x_{2n}, x_{2n-1}) \leq H_d(Tx_{2n-2}, Sx_{2n-1}) \quad \text{for } n = 1, 2, 3, \ldots. \tag{5}$$

For any natural numbers $m > n > \max\{n_0, n_1\}$, from (2) (or (3)) and the inequality (4) (or (5)) it follows

$$\begin{aligned} f(d(x_{n+1}, x_n)) &\leq f(H_d(Tx_n, Sx_{n-1}))(\text{or } f(H_d(Tx_{n-1}, Sx_n))) \\ &\leq f(M(x_n, x_{n-1})) - \varphi(f(M(x_n, x_{n-1}))) \\ &\leq (1-a)f(M(x_n, x_{n-1})) = (1-a)f(d(x_n, x_{n-1})). \end{aligned}$$

By this inequality, we get

$$f(d(x_k, x_{k-1})) \leq (1-a)f(d(x_{k-1}, x_{k-2})) \leq \cdots \leq (1-a)^{k-n} f(d(x_n, x_{n-1})) \quad \text{for } k > n.$$

Therefore, from the assumption of f we have

$$\begin{aligned} f(d(x_m, x_n)) &\leq f(d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)) \\ &\leq f(d(x_m, x_{m-1})) + f(d(x_{m-1}, x_{m-2})) + \cdots + f(d(x_{n+1}, x_n)) \\ &\leq (1-a)^{m-n} f(d(x_n, x_{n-1})) + (1-a)^{m-n-1} f(d(x_n, x_{n-1})) + \cdots + (1-a)f(d(x_n, x_{n-1})) \\ &= \frac{(1-a) - (1-a)^{m-n+1}}{1-(1-a)} f(d(x_n, x_{n-1})) \\ &< \frac{1-a}{a} f(d(x_n, x_{n-1})) \leq \frac{1-a}{a} f(\delta) < f(\varepsilon). \end{aligned}$$

This shows that $d(x_m, x_n) < \varepsilon$. Since ε is arbitrary, $\{x_n\}$ is a Cauchy sequence. By means of the completeness of E we infer that the sequence $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = x^*$ with $x^* \in E$.

Now we prove $d(Tx^*, x^*) = 0$. Suppose that this is not true, then $d(Tx^*, x^*) > 0$. For large enough n, we claim that the following equation holds

$$M(x^*, x_{2n+1}) = \max\left\{d(x^*, x_{2n+1}), d(Tx^*, x^*), d(Sx_{2n+1}, x_{2n+1}), \frac{1}{2}(d(Tx^*, x_{2n+1}) + d(Sx_{2n+1}, x^*))\right\} = d(Tx^*, x^*).$$

Indeed, since $\lim_{n\to\infty} d(x^*, x_{2n+1}) = 0$ and $\lim_{n\to\infty} d(Sx_{2n+1}, x_{2n+1}) = 0$, it follows that

$$\begin{aligned}\lim_{n\to\infty}\left[(d(Tx^*, x_{2n+1}) + d(Sx_{2n+1}, x^*))\right] &\le \lim_{n\to\infty}[(d(Tx^*, x^*) + d(x^*, x_{2n+1}) + d(Sx_{2n+1}, x_{2n+1}) + d(x_{2n+1}, x^*))]\\ &= d(Tx^*, x^*).\end{aligned}$$

Therefore, there exists n_1 such that $M(x^*, x_{2n+1}) = d(Tx^*, x^*)$ for $\forall n > n_1$. Note that

$$f(d(Tx^*, x_{2n+2})) \le f(H_d(Tx^*, Sx_{2n+1})) \le f(M(x^*, x_{2n+1})) - \varphi(f(M(x^*, x_{2n+1}))).$$

Let $n \to \infty$ and apply (H1) and the properties of f and φ, we get

$$f(d(Tx^*, x^*)) \le f(d(Tx^*, x^*)) - \varphi(f(d(Tx^*, x^*))) < f(d(Tx^*, x^*)).$$

This is a contradiction. So $d(Tx^*, x^*) = 0$, in virtue of the approximation of Tx^*, we have $x^* \in Tx^*$. The above proof also guarantees $x^* \in Sx^*$. This completes the proof of Theorem 3.1. $\square$

Similarly, we have

Theorem 3.2. *Let the condition* (H1) *hold and the multivalued mappings T and S both have LCAV and satisfy weakly generalized contraction with respect to given $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a-0))$. Then T and S admit a common fixed point. Further, the iterated convergence of Theorem 3.1 holds.*

Corollary 3.3. *Under the assumptions of Theorem 3.1 (resp. under the assumptions of Theorem 3.2), suppose that T and S both are single-valued operators and satisfy*

$$f(d(Tx, Sy)) \le f(M(x, y)) - \varphi(f(M(x, y)))$$

for $f \in \mathscr{F}[0, a)$, $\varphi \in \Phi[0, a)$ and each $x, y \in E$, where

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}(d(Tx, y) + d(Sy, x))\right\}.$$

Then T, S have a unique common fixed point $x^* \in E$. Further, for each $x_0 \in E$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T and S.

Proof. Theorem 3.1 (resp. Theorem 3.2) ensures existence of common fixed points. To prove the uniqueness, let y^* be any common fixed point of T and S. If $x^* \ne y^*$, then $d(x^*, y^*) > 0$. Thus,

$$M(x^*, y^*) = \max\left\{d(x^*, y^*), d(Tx^*, x^*), d(Sy^*, y^*), \frac{1}{2}(d(Tx^*, y^*) + d(Sy^*, x^*))\right\} = d(x^*, y^*).$$

This yields

$$f(d(x^*, y^*)) = f(d(Tx^*, Sy^*)) \le f(M(x^*, y^*)) - \varphi(f(M(x^*, y^*))) < f(M(x^*, y^*)) = f(d(x^*, y^*)),$$

a contradiction, so $d(x^*, y^*) = 0$, i.e. $x^* = y^*$. $\square$

In what follows, we present the uniqueness of common fixed points for multivalued mappings.

Theorem 3.4. *Let E be an totally ordered complete metric space and let E satisfy* (H1) *and the following*

(H2) *$x \le y \le z$ implies that $d(z, x) \ge d(y, x)$ for all $x, y, z \in E$.*

Suppose that T and S satisfy all conditions given in Theorem 3.1 (resp. in Theorem 3.2), then T, S have a unique common fixed point $x^* \in E$ ant the iterated convergence of Theorem 3.1 holds.

Proof. Theorem 3.1 (resp. Theorem 3.2) ensures existence of common fixed points. To prove the uniqueness, let both x^* and y^* be common fixed points of T and S. Because $(E, \preceq)$ is a totally ordered space, we have either $x^* > y^*$ or $y^* > x^*$. Without loss of generality, we assume that the former is true. If T has UCAV, we have $x \in Tx^*$ with $x \ge x^*$ and $d(x, y^*) = d(Tx^*, y^*)$. From our assumption it follows that $d(x, y^*) \ge d(x^*, y^*)$. On the other hand, $x^* \in Tx^*$ implies that $d(x, y^*) \le d(x^*, y^*)$. Hence,

$$d(x, y^*) = d(x^*, y^*) = d(Tx^*, y^*).$$

If $x^* \neq y^*$, then $d(x^*, y^*) > 0$. Thus

$$d(x^*, y^*) = d(Tx^*, y^*) \leq H_d(Tx^*, Sy^*). \tag{6}$$

If T has LCAV, so does S, we have $y \in Sy^*$ with $y \leq y^*$ and $d(y, x^*) = d(Sy^*, x^*)$. From (H2) it follows that $d(y, x^*) \geq d(x^*, y^*)$. On the other hand, $y^* \in Sy^*$ implies that $d(y, x^*) \leq d(x^*, y^*)$. Hence,

$$d(y, x^*) = d(x^*, y^*) = d(x^*, Sy^*).$$

At all events, (6) holds if $x^* \neq y^*$. We observe that

$$M(x^*, y^*) = \max\left\{d(x^*, y^*), d(Tx^*, x^*), d(Sy^*, y^*), \frac{1}{2}(d(Tx^*, y^*) + d(Sy^*, x^*))\right\} = d(x^*, y^*),$$

it yields

$$f(d(x^*, y^*)) \leq f(H_d(Tx^*, Sy^*)) \leq f(M(x^*, y^*)) - \varphi(f(M(x^*, y^*))) < f(d(x^*, y^*)).$$

This is a contradiction. Consequently, the inequality $x^* < y^*$ is not true. By the same methods we can verify that $y^* < x^*$ is also not true. Thus $x^* = y^*$. □

Theorem 3.5. *Let E satisfy the hypothesis* (H1). *Suppose that the multivalued mapping T has UCAV and satisfies the weakly generalized contraction with respect to given $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a - 0))$, then T have a fixed point $x^* \in E$. Further, for each $x_0 \in E$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T.*

In addition, assume that, besides the above assumptions, (H2) holds, then T has a unique fixed point.

Remark 3.6. The results of Corollary 3.3 and Theorem 3.5 are the extension and improvement of the corresponding results of [30,16], respectively.

Theorem 3.7. *Let the condition* (H1) *hold and the multivalued mappings T and S have weakly generalized contraction with respect to given $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a - 0))$. If the following hypothesis holds, the T and S admit a common fixed point. Further, the iterated convergence of Theorem* 3.1 *holds.*

- *T and S both have AV, are nondecreasing and there exists $x_0 \in E$ such that $\{x_0\} \leq Tx_0 \leq Sx_0$.*

Proof. If $x_0 \in Tx_0$, then the proof is finished. Otherwise, for any $x \in Tu_0$ one has that $x \geq x_0$. Since T has approximative values, there exists $x_1 \in Tx_0$ with $x_1 \geq x_0$ and $d(x_0, x_1) = d(Tx_0, x_0)$. We now have $x \geq x_1$ for all $x \in Sx_1$. If $x_1 \in Sx_1$, the proof is finished. Otherwise, by means of the fact that S has AV, there exists $x_2 \in Sx_1$ with $x_2 \geq x_1$ and $d(x_2, x_1) = d(Sx_1, x_1)$. We continue the procedure of constructing x_n inductively, that is, either $x_{2n+1} \in Tx_{2n}$ or $x_{2n+2} \in Sx_{2n+1}$, then our proof is complete; or there exists $x_{2n+1} \in Tx_{2n}$ with $x_{2n+1} \neq x_{2n}$ and $x_{2n+1} \geq x_{2n}$, also, $x_{2n+2} \in Sx_{2n+1}$ with $x_{2n+2} \neq x_{2n+1}$ and $x_{2n+2} \geq x_{2n+1}$, where $n = 0, 1, 2, \ldots$, such that (1), (2) and (3) hold. The rest of this proof is the same as that of Theorem 3.1. This proof is complete. □

4. Examples

Example 4.1. Let $E = [0, \frac{\pi}{4}] \times [0, \frac{\pi}{4}]$ and the metric d be the Euclidean, i.e.

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We define the partial order in E by $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$ for all $X = (x_1, x_2)$, $Y = (y_1, y_2) \in E$. Obviously, E is a complete metric space. Suppose T and S are multivalued operators $E \to 2^E$ defined by

$$SP = TP = \left[\frac{1}{2}\sin x \cos x, \frac{1}{2}\sin x\right] \times \left[\frac{1}{2}\sin y \cos y, \frac{1}{2}\sin y\right]$$

with $P = (x, y) \in E$. Now let us check that the conditions of Theorem 3.1 are satisfied. T and S have UCAV in E. Trivial. For all $X = (x_1, x_2)$, $Y = (y_1, y_2) \in E$, it is easy to see

$$H_d(TX, SY) \leq \frac{1}{2}d(X, Y).$$

Taking $f(t) = t$, $\varphi(t) = \frac{1}{2}t$, then $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a - 0))$ and $f(H_d(TX, SY)) \leq f(M(X, Y)) - \varphi(f(M(X, Y)))$. Therefore the multivalued operators T and S satisfy all conditions of Theorem 3.1. Consequently, T, S admit at least a common fixed point.

Example 4.2. Let $E = \{x \in C[a, b] : x(t) \geq 0 \text{ for } t \in [a, b]\}$ and the metric d be defined by

$$d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|.$$

Obviously E is a complete metric space. The partial order of E is defined by $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [a, b]$. Let $f(t) = t$ and $\varphi(t) = \frac{1}{3}t$, then $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a - 0))$. Suppose T and S are multivalued operators from E into 2^E defined by

$$(Sx)(t) = \left[\frac{x(t)}{4}, \frac{2x(t)}{5}\right], \qquad (Tx)(t) = \left[\frac{x(t)}{8}, \frac{3x(t)}{8}\right], \quad t \in [a, b].$$

Now let us check all conditions of Theorem 3.2. Clearly, the multivalued operators T and S both have LCAV in E. For $\forall x, y \in E$ we have

$$\begin{aligned} f(H_d(Tx(t), Sy(t))) &\leq \max\left(\frac{3x(t)}{8}, \frac{2y(t)}{5}\right) \\ &\leq \frac{2}{3}\max\{d(Tx(t), x(t)), d(Sy(t), y(t))\} \\ &\leq \frac{2}{3}M(x(t), y(t)) = f(M(x, y)) - \varphi(f(M(x, y))). \end{aligned}$$

Therefore, an application of Theorem 3.2 yields that the multivalued operators T and S have the common fixed point $x^* \equiv 0$.

Example 4.3. Let $E = [0, +\infty)$ and the metric be defined by

$$d(x, y) = |x - y|.$$

Obviously, E is a totally ordered complete metric space and satisfies (H1) and (H2). Suppose T and S are multivalued operators $E \to 2^E$ defined by

$$Sx = Tx = \left[\frac{x}{4}, \frac{x}{2}\right].$$

Now let us check that the conditions of Theorem 3.4 are satisfied. T and S have LCAV in E. Trivial. For all $x, y \in E$, it is easy to see

$$H_d(Tx, Sy) \leq \frac{1}{2}|x - y| = \frac{1}{2}d(x, y).$$

Taking $f(t) = t, \varphi(t) = \frac{1}{2}t$, then $f \in \mathscr{F}(0, a)$ and $\varphi \in \Phi[0, f(a - 0))$ and $f(H_d(Tx, Sy)) \leq f(M(x, y)) - \varphi(f(M(x, y)))$. Therefore the multivalued operators T and S satisfy all conditions of Theorem 3.4. Consequently, T and S have a unique common fixed point.

Example 4.4. Let $\psi : R^+ \to R$ satisfy that

(i) ψ is nonnegative and Lebesgue integrable.
(ii) $\int_0^\varepsilon \psi(t)dt > 0$ for each $\varepsilon > 0$.

Let $f_1(t) = \int_0^t \psi(s)ds > 0$. It is easy to see that $f_1 \in \mathscr{F}(0, a)$. Let

$$\varphi_1(t) = \begin{cases} \frac{1}{2}t, & t \in [0, 1], \\ \frac{1}{3}t, & t \in (1, 2], \\ \cdots, & \\ \frac{1}{n}t, & t \in (n - 2, n - 1], \\ \cdots. & \end{cases}$$

Clearly, $\varphi_1 \in \Phi[0, f_1(a - 0))$.

Conclusion: Suppose that E is an ordered complete metric space and satisfies (H1), multivalued mappings T and S have UCAV (resp. LCAV) and satisfy

$$f_1(H_d(Tx, Sy)) \leq f_1(M(x, y)) - \varphi_1(f(M(x, y)))$$

for each $x, y \in E$, then T, S have a common fixed point $x^* \in E$.

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